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Modified Exponential-Gamma Distribution Using Metropolis-Hastings Algorithm

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Abstract: This study introduces the Modified Exponential-Gamma (ME-G) distribution, a novel one-parameter lifetime model constructed as a mixture of an exponential and a gamma distribution with fixed shape parameter 7. Motivated by the limitations of existing one-parameter models in capturing complex lifetime behaviours, the ME-G distribution aims to provide enhanced flexibility and improved fit for reliability and survival data. Key statistical properties, including probability density, survival, hazard, and moment functions, are derived and examined. Parameter estimation is performed using both Maximum Likelihood Estimation (MLE) and the Metropolis-Hastings (M-H) algorithm, with the M-H algorithm demonstrating superior goodness-of-fit performance. Comparative analyses against established one-parameter distributions such as Suja, Rama, Akash, Lindley, and Exponential, using real aircraft window glass strength data and simulated datasets confirm the ME-G distribution's superior fit as indicated by lower information criterion scores. The results highlight the ME-G distribution's potential as a robust alternative for lifetime modelling, especially when conventional models fall short. This work advances lifetime distribution theory by combining analytical tractability with practical efficacy, offering a promising tool for future reliability and survival analysis.

Keywords: Lifetime distribution, Moments, Hazard rate function, Survival function, Stress-strength reliability, Metropolis-Hasting algorithm.

Subject Classification: 62E15, 65C05.

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1 Introduction

Modelling and analysing lifetime data remains a central task in numerous applied fields such as biomedicine, engineering, insurance, and finance, where understanding time-to-event phenomena is critical. Classical lifetime distributions, notably the Exponential and Lindley distributions, have been widely adopted due to their simplicity and ease of interpretation (Lindley, 1958). However, these models often fall short in capturing the diverse and complex behaviours observed in real-world lifetime data, particularly in their inability to flexibly model varying hazard rate structures (Shanker *et al.,* 2015). In response, several one-parameter extensions have been proposed to improve modelling accuracy and adaptability. Among these, the Akash, Shanker, Aradhana, Amarendra, Sujatha, Rama, and Suja distributions introduced by Shanker (2015, 2016, 2017) have demonstrated enhanced performance over traditional models. Nevertheless, these distributions still exhibit limitations in adequately fitting datasets with more intricate hazard rate patterns or skewness.

The shortcomings of existing one-parameter lifetime models, namely, limited flexibility in hazard rate behaviour and suboptimal fit to heterogeneous data serve as the primary motivation for this study. Addressing this gap, the study proposes a new one-parameter model: the modified Exponential-Gamma (ME-G) distribution, which builds upon and extends the foundational work of Lindley (1958) and Shanker (2015). The ME-G distribution is derived from a mixture framework that incorporates a Gamma distribution with an increased shape parameter of 7, thereby broadening the distribution's flexibility to accommodate a wider range of hazard functions.

Several authors have proposed distributions similar, though distinct, to the ME-G distribution. Rasekhi *et al.,* (2017) introduced the modified Exponential distribution with applications, a four-parameter model defined by $(\alpha, \beta, \gamma, \lambda)$. Suleman *et al.,* (2025) proposed a four-parameter distribution $(\alpha, \beta, \theta, \lambda)$ titled the notes on a modified Exponential-Gamma Distribution: Its Properties and Applications. Umar *et al.,* (2021), whose work is most closely related yet different from the ME-G distribution, developed the new Exponential-Gamma distribution with applications, involving two parameters (α, θ). Unlike the present study, which provides an in-depth investigation of various statistical properties of the ME-G model including the survival function, hazard rate function, mean residual life function, stochastic orderings, moments, Bonferroni and Lorenz curves, and stress-strength reliability, these prior studies offer limited theoretical analysis. Furthermore, neither Umar *et al.,* (2021) nor Suleman *et al.,* (2025) included simulation-based empirical analysis, which is a key component of the current research.

The principal objective of this research is to introduce the ME-G distribution and investigate its theoretical and statistical properties, including its probability density function (PDF), survival function, hazard rate function, moments, entropy, stochastic orderings, and measures such as Bonferroni and Lorenz curves, and stress-strength reliability. Through this in-depth analysis, the study aims to demonstrate that the ME-G distribution provides a more robust and adaptable model for lifetime data.

Furthermore, the study emphasises robust parameter estimation, recognising that accurate inference depends heavily on estimation technique. The study adopts the Metropolis-Hastings (M-H) algorithm, a Markov Chain Monte Carlo (MCMC) method, given its effectiveness in parameter estimation, especially in scenarios where classical maximum likelihood estimation (MLE) may be inadequate or computationally burdensome. Comparative analysis between M-H and MLE methods is undertaken to evaluate their respective efficiencies in estimating the parameters of the ME-G distribution.

To validate the practical applicability of the proposed model, the ME-G distribution is fitted to both simulated and real lifetime datasets. Its performance is benchmarked against existing one-parameter models to assess its superiority in terms of goodness-of-fit and modelling accuracy.

In summary, this study introduces a novel lifetime distribution designed to overcome the rigidity of traditional models and offers a comprehensive estimation strategy. By addressing critical limitations in the current literature, the ME-G distribution advances the field of lifetime modelling and provides a valuable tool for practitioners and researchers in diverse scientific domains.

2 Methodologies

2.1 A Modified Exponential-Gamma (ME-G) Distribution

2.1.1 Probability Density Function (PDF) and Cumulative Density Function (CDF)

The proposed novel one-parameter lifetime distribution is formulated using a two-component mixture model. This mixture consists of an exponential distribution with a scale parameter θ and a gamma distribution with a shape parameter of 7 and a scale parameter also denoted as θ. The combination of these two distributions is regulated by their respective mixing proportions, given by $\frac{\theta^6}{\theta^6+720}$ and $\frac{720}{\theta^6+720}$.

The PDF of the modified Exponential-Gamma (ME-G) distribution is therefore generated using the formular below:

$$f(x;\theta) = p_1 g_1 + p_2 g_2 \tag{2.1}$$

Where:

$$p_1 = \frac{\theta^6}{\theta^6 + 720},\ g_1 \sim Exp(\theta),\ p_2 = \frac{720}{\theta^6 + 720},\ \text{and } g_2 \sim Gamma(7;\theta)$$

The Cumulative Density Function (CDF) of the modified Exponential-Gamma (ME-G) distribution is derived using the mathematical formular below:

$$F(x;\theta) = \int_{\infty}^{x} f(x;\theta)\, dx \tag{2.2}$$

Where $f(x;\theta)$ is the PDF of the modified Exponential-Gamma (ME-G) distribution.

2.2 Properties of the Modified Exponential-Gamma (ME-G) Distribution

2.2.1 Moments

The moment generating function corresponding to the modified Exponential-Gamma (ME-G) distribution, as defined in equation (2.1), can be derived as follows:

$$M_x(t) = \int_0^{\infty} f(x;\theta)\, e^{(tx)}\, dx \tag{2.3}$$

2.2.2 Survival Rate Function, Hazard Rate Function, and Mean Residual Life Function

Survival rate function, hazard rate function, and mean residual life function are fundamental concepts in reliability analysis, survival analysis, and lifetime data modelling. Survival rate function, also known as survival function, is used to describe the probability that a system, individual, or component survives beyond a given time (Chisimkwuo *et al.,* 2022; Suleman *et al.,* 2025). Hazard rate function, also known as the failure rate function, describes the instantaneous failure rate of a system or individual at a given time, given that it has survived up to that time (Gemeay *et al.,* 2024). Mean residual life function, also known as the expected residual lifetime function, quantifies the expected remaining lifetime given that a system, individual, or component has already survived up to a certain time (Pokalas *et al.,* 2021). Survival rate function, hazard rate function, and mean residual life function denoted as $s(x)$, $h(x)$, and $m(x)$ are defined respectively as follows:

$$s(x) = 1 - F(x) \tag{2.4}$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{f(x)}{1 - F(x)} \tag{2.5}$$

$$m(x) = \frac{1}{s(x)} \int_x^{\infty} s(t)\, dt \tag{2.6}$$

2.2.3 Stochastic Orderings

Stochastic ordering is a fundamental concept in probability and distribution theory, used to compare random variables and probability distributions based on their likelihood of taking larger values (Chisimkwuo *et al.,* 2024; Pokalas *et al.,* 2024). It plays a crucial role in various fields such as reliability theory, risk analysis, actuarial science, and economics (Okereke *et al.,* 2021). This research studied four stochastic orderings which are stochastic order, hazard rate order, mean residual life order, and likelihood ratio order.

2.2.4 Entropy Measures

The entropy of a random variable X signifies a measure of the variation in uncertainty. A widely recognized measure of entropy is the Rényi entropy (Rényi, 1960). Assuming X is a continuous random variable with a probability density function denoted as $f(.)$, the definition of Renyi entropy is as follows:

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\} \tag{2.7}$$

Where:

$\gamma > 0$ and $\gamma \neq 1$.

2.2.5 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves, introduced by (Bonferroni, 1930), along with the Bonferroni and Gini indices, have found applications across many fields. While economics commonly uses them to study income disparities and poverty, their utility extends to other domains such as reliability, demography, insurance, and medicine. The definitions of the Bonferroni and Lorenz curves are derived as follows:

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x)\, dx \tag{2.8}$$

And

$$L(p) = \frac{1}{\mu} \int_0^q x f(x)\, dx \tag{2.9}$$

respectively.

Where $\mu = E(X)$ and $q = F^{-1}(p)$.

2.2.6 Stress-Strength Reliability

According to Okereke and Uwaeme (Okereke & Uwaeme, 2018), the concept of stress-strength reliability pertains to the lifespan of a component with a random strength variable X, which is subjected to a random stress variable Y. The component experiences immediate failure when the applied stress surpasses its strength, and it will continue to operate satisfactorily if X exceeds Y. Consequently, $R = P(X < Y)$ serves as a measure of component reliability and is referred to as the stress-strength parameter in statistical literature. This concept finds extensive applications across a broad spectrum of knowledge areas, particularly in engineering fields such as structural engineering, rocket motor deterioration, ceramic component static fatigue, concrete pressure vessel ageing, and more. The stress-strength reliability is derived as follows:

$$R = P(X < Y) = \int_0^\infty P[Y < X \mid X = x]\, f_x(x)\, dx \tag{2.10}$$

2.3 Parameter Estimation of the Modified Exponential-Gamma (ME-G) Distribution

2.3.1 Maximum Likelihood Estimation (MLE)

Maximum likelihood estimation (MLE) is a fundamental statistical method used to estimate the parameters of a probability distribution by maximizing the likelihood function (Lauritzen et *al.,* 2019). In distribution theory, MLE plays a crucial role in fitting probability models to data by determining the parameter values that make the observed data most probable. Let $L(x_1, x_2, \ldots, x_n;\ \theta)$ be a random sample from the modified Exponential-Gamma (ME-G) distribution, then the maximum likelihood estimation is derived as follows:

$$L\left(x_1, x_2, \ldots, x_n; \theta\right) = \prod_{i=1}^{n} \left[f\left(x_i; \theta\right)\right] \tag{2.11}$$

2.3.2 Metropolis-Hastings (M-H) Algorithm

The Metropolis-Hastings (M-H) algorithm is a Markov Chain Monte Carlo (MCMC) method used for parameter estimation in probability distributions, particularly when the likelihood function is complex or analytically intractable (Yan-ling *et al.,* 2011). This study employs the MH algorithm to estimate the parameter of the modified Exponential-Gamma (ME-G) distribution by generating posterior samples and optimizing the likelihood function. The M-H algorithm is implemented as follows:

1. Define the Target Distribution: The target distribution is the modified Exponential-Gamma (ME-G) distribution and is derived as seen in (2.1).
2. Initialization**:** An initial value x_0 is randomly selected from a Uniform distribution over the range of the data.
3. Proposal Step: A new candidate sample x^* is drawn from a normal distribution centred at the current sample x_{i-1}.

$$x^* \sim N(x_{i-1}, \sigma^2) \tag{2.12}$$

Where σ^2 is the proposal standard deviation.

4. Acceptance Criterion: The candidate x^* is accepted with probability:

$$r = min\left(\frac{f(x^*|\theta)}{f(x_{i-1}|\theta)}\right) \tag{2.13}$$

Accept x^* with probability $\min(1, r)$; otherwise, the current sample is retained.

5. Iteration: Steps 3 and 4 are repeated for a predefined number of iterations (n).
6. Output: The generated sequence of samples approximates the posterior distribution of θ.
7. Estimate the Parameter θ: To estimate θ, a likelihood-based optimization procedure is applied to the M-H generated samples. The likelihood functions for a given set of observed samples $\{x_i\}_{i=1}^{n}$:

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) \tag{2.14}$$

Taking the logarithm, the log-likelihood function is:

$$L(\theta) = \sum_{i=1}^{n} \log f(x_i|\theta) \tag{2.15}$$

To estimate θ, the negative log-likelihood function is minimized using numerical optimization methods, specifically the Brent optimization technique. The optimization problem is defined as:

$$\hat{\theta} = arg\ \min_{\theta}\left(-\sum_{i=1}^{n} \log f(x_i|\theta)\right) \tag{2.16}$$

2.4 Comparative Analysis

This study examines the applicability of the modified Exponential-Gamma (ME-G) distribution by evaluating its goodness of fit using lifetime data obtained from an engineering study conducted by Fuller *et al.,* (1994) and a simulated data set generated using M-H algorithm above. The performance of the modified Exponential-Gamma (ME-G) distribution is subsequently compared with that of other one-parameter lifetime distributions, including the Suja, Rama, Akash, Lindley, and Exponential distributions. To facilitate a comprehensive comparison, various statistical criteria are employed, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC). The mathematical formulations for computing these model selection criteria are provided as follows:

$$\mathrm{AIC} = -2\log L + 2k \quad (2.17)$$

$$\mathrm{BIC} = -2\log L + \log(n)\,k \quad (2.18)$$

$$\mathrm{CAIC} = -2\log_e\left[\mathrm{L}(\hat{\theta})\right] + \mathrm{k}\left[\log_e(\mathrm{n}) + 1\right] \quad (2.19)$$

$$\mathrm{HQIC} = -2\mathrm{L}_{\mathrm{max}} + 2\mathrm{k}\log(\log(\mathrm{n})) \quad (2.20)$$

Where:

$\log L$ and $\mathrm{L}_{\mathrm{max}}$ are the maximized value of the log-likelihood function under the distribution considered, n is the sample size, and k is the number of parameters.

The best distribution is the distribution which corresponds to lower values of AIC, BIC, CAIC, and HQIC.

3 Results and Discussion

3.1 A Modified Exponential-Gamma (ME-G) Distribution

3.1.1 Probability Density Function (PDF) and Cumulative Density Function (CDF)

The PDF of modified Exponential-Gamma (ME-G) distribution from equation (2.1) is formally presented as follows:

$$f(x;\theta) = \frac{\theta^7}{\theta^6 + 720}\left(1 + x^6\right) e^{-\theta x} \quad ; x > 0, \theta > 0 \tag{3.1}$$

The CDF corresponding to equation (3.1) can be derived as follows:

$$F(x;\theta) = 1 - \left[1 + \frac{\theta^6 x^6 + 6\theta^5 x^5 + 30\theta^4 x^4 + 120\theta^3 x^3 + 360\theta^2 x^2 + 720\theta x}{\theta^6 + 720}\right] e^{-\theta x}; \; x, \theta > 0 \tag{3.2}$$

A graphical representation of the PDF and CDF for the modified Exponential-Gamma (ME-G) distribution for different values of θ are represented in Figures 1 and 2 respectively.

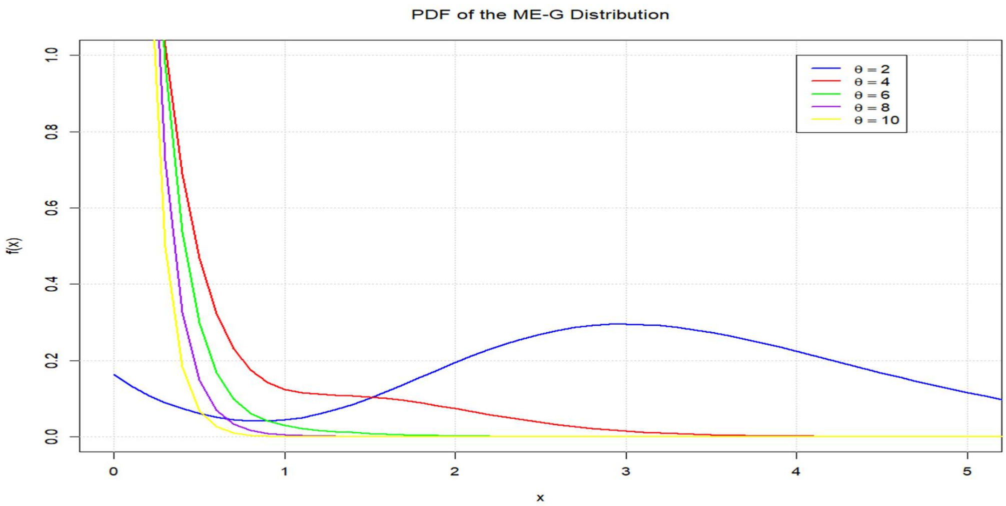

Figure 1: *Graph of the PDF of the ME-G distribution for varying values of the parameter θ.*

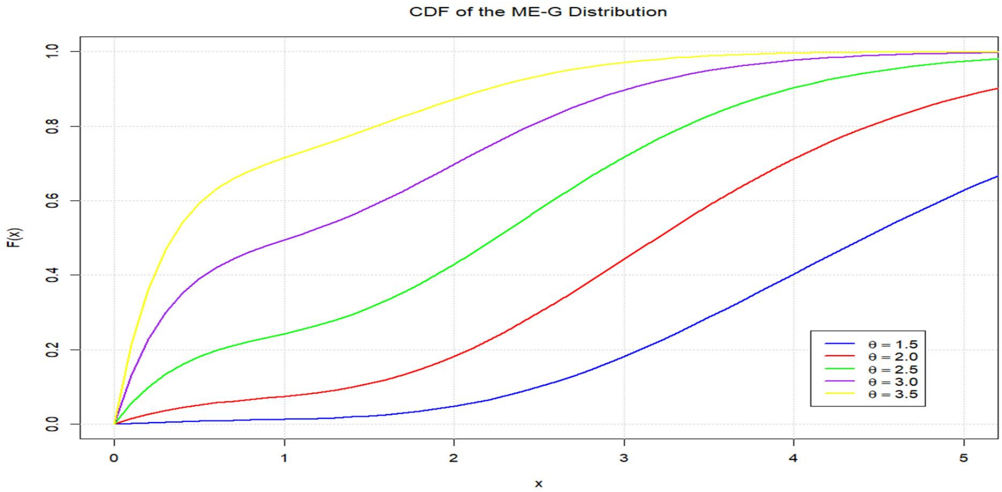

Figure 2: *Graph of the CDF of the ME-G distribution for varying values of the parameter θ.*

3.2 Properties of the Modified Exponential-Gamma (ME-G) Distribution

3.2.1 Moments

The moment generating function corresponding to the modified Exponential-Gamma (ME-G) distribution, as defined in equation (3.1), can be derived as follows:

$$M_x(t) = \frac{\theta^7}{\theta^6 + 720}\int_0^\infty e^{-(\theta - t)x}\left(1 + x^6\right)dx$$

$$= \frac{\theta^7}{\theta^6 + 720}\left[\frac{1}{(\theta - t)} + \frac{720}{(\theta - t)^7}\right]$$

$$= \frac{\theta^7}{\theta^6 + 720}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k + \frac{720}{\theta^7}\sum_{k=0}^{\infty}\binom{k+6}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$= \sum_{k=0}^{\infty}\frac{\theta^6(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}{\theta^6 + 720}\left(\frac{t}{\theta}\right)^k \quad (3.3)$$

Consequently, the r moment about the origin of the modified Exponential-Gamma (ME-G) distribution, as outlined in equation (3.1), can be computed as follows:

$$\mu_r' = \frac{r!\left[\theta^6 + (r+1)(r+2)(r+3)(r+4)(r+5)(r+6)\right]}{\theta^r\left(\theta^6+720\right)}; r=1,2,3,\ldots \quad (3.4)$$

The first four moments about the origin of modified Exponential-Gamma (ME-G) distribution (3.1) are as follows:

$$\mu_1' = \frac{\theta^6+5040}{\theta\left(\theta^6+720\right)}, \quad \mu_2' = \frac{2(\theta^6+201600)}{\theta^2\left(\theta^6+720\right)}, \quad \mu_3' = \frac{6(\theta^6+60480)}{\theta^3\left(\theta^6+720\right)}, \quad \mu_4' = \frac{24(\theta^6+151200)}{\theta^4\left(\theta^6+720\right)}$$

3.2.2 Survival Rate Function, Hazard Rate Function, and Mean Residual Life Function

The associated survival function $s(x)$, hazard rate function $h(x)$, and mean residual life function $m(x)$ corresponding to the modified Exponential-Gamma (ME-G) distribution can therefore be derived as follows:

$$s(x) = \left[1 + \frac{\theta^6x^6 + 6\theta^5x^5 + 30\theta^4x^4 + 120\theta^3x^3 + 360\theta^2x^2 + 720\theta x}{\theta^6+720}\right]e^{-\theta x} \quad (3.5)$$

$$h(x) = \frac{\theta^7(1+x^6)}{\theta^6(x^6+1) + 6\theta^5x^5 + 30\theta^4x^4 + 120\theta^3x^3 + 360\theta^2x^2 + 720(\theta x+1)} \quad (3.6)$$

And

$$m(x) = \frac{1}{\left[\theta^6\left(x^6+1\right) + 6\theta^5x^5 + 30\theta^4x^4 + 120\theta^3x^3 + 360\theta^2x^2 + 720(\theta x+1)\right]e^{-\theta x}} \quad (3.7)$$

$$\int_x^\infty \left[\theta^6\left(t^6+1\right) + 6\theta^5t^5 + 30\theta^4t^4 + 120\theta^3t^3 + 360\theta^2t^2 + 720(\theta t+1)\right]e^{-\theta t}dt$$

$$= \frac{\theta^6\left(x^6+1\right) + 12\theta^5x^5 + 90\theta^4x^4 + 480\theta^3x^3 + 1800\theta^2x^2 + 4320\theta x + 5040}{\theta\left[\theta^6\left(x^6+1\right) + 6\theta^5x^5 + 30\theta^4x^4 + 120\theta^3x^3 + 360\theta^2x^2 + 720(\theta x+1)\right]} \quad (3.8)$$

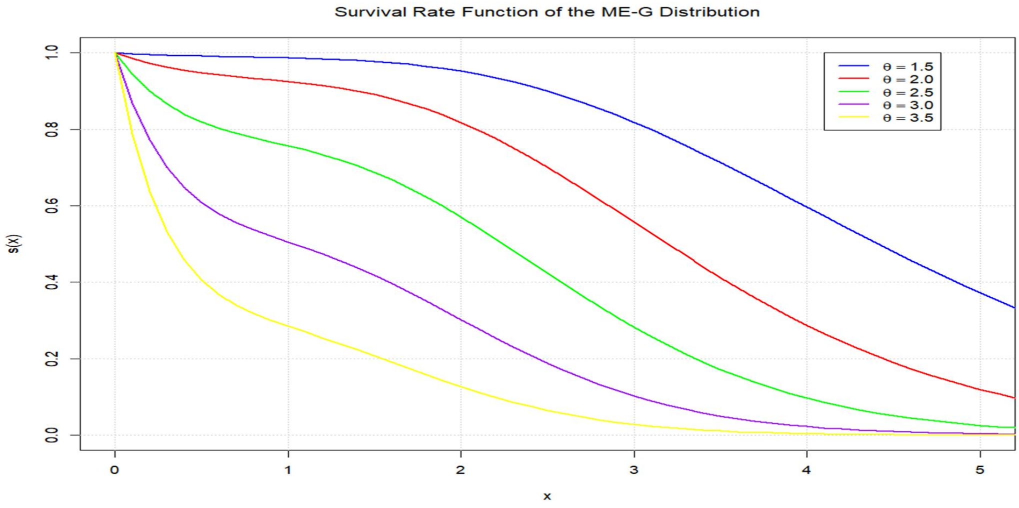

Figure 3: *Graph of the survival function of the ME-G distribution for different values of the parameter θ.*

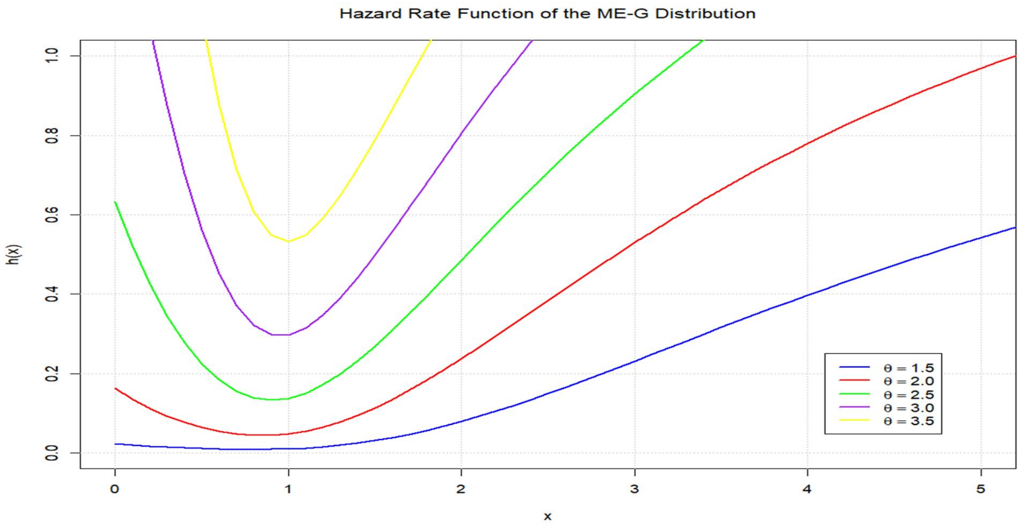

Figure 4: *Graph of hazard rate function of the ME-G distribution for different values of the parameter θ.*

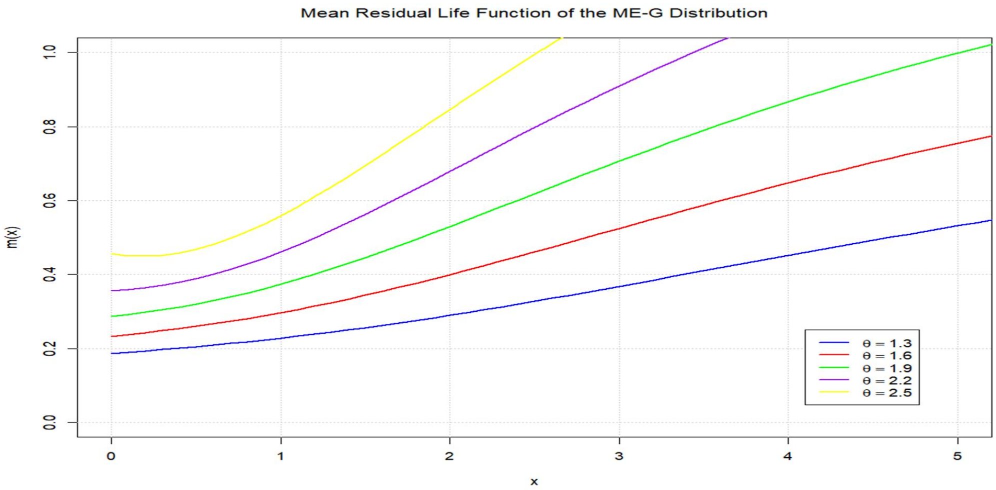

Figure 5: *Graph of mean residual life function of the ME-G distribution for different values of the parameter θ.*

It can be easily verified that:

$$h(0) = \frac{\theta^7}{\theta^6 + 720} = f(0) \quad and \quad m(0) = \frac{\theta^6 + 5040}{\theta(\theta^6 + 720)} = \mu_1'.$$

The graphical illustrations of the survival function, hazard rate, and mean residual life function of the modified Exponential-Gamma (ME-G) distribution across a spectrum of parameter values are represented in Figures 3, 4, and 5, respectively.

3.2.3 Stochastic Orderings

The stochastic ordering of positive continuous random variables is a significant instrument for assessing their comparative behaviors. According to Shanker (2015), a random variable X is deemed to be less than a random variable Y in the context of the following conditions:

i. Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
ii. Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.

iii. Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x.

iv. Likelihood ratio order $(X \leq_{lr} Y)$ if $\left(\frac{f_X(x)}{f_Y(x)}\right)$ decreases in x.

The subsequent findings, credited to (Shaked *et al.,* 1994), are widely recognized for their role in determining the stochastic ordering of distributions:

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y)$$

$$\Downarrow$$

$$(X \leq_{st} Y) \tag{3.9}$$

The modified Exponential-Gamma (ME-G) distribution follows the most robust likelihood ratio ordering, as demonstrated in the subsequent theorem.

Theorem: Let $X \sim ME - G\ distribution(\theta_1)$ and $Y \sim ME - G\ distribution(\theta_2)$.

If $\theta_1 > \theta_2$, then $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y), (X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$

Proof: This gives:

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^7(\theta_2^6 + 720)}{\theta_2^7(\theta_1^6 + 720)} e^{-(\theta_1 + \theta_2)}; \quad x > 0$$

Now

$$ln\frac{f_X(x)}{f_Y(x)} = ln\left[\frac{\theta_1^7(\theta_2^6 + 720)}{\theta_2^7(\theta_1^6 + 720)}\right] - (\theta_1 + \theta_2)$$

This gives.

$$\frac{d}{dx} ln\frac{f_X(x)}{f_Y(x)} = -(\theta_1 + \theta_2)$$

Thus for $\theta_1 > \theta_2$, $\frac{d}{dx} ln \frac{f_X(x)}{f_Y(x)} < 0$, this means that $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y), (X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$.

3.2.4 Entropy Measures

The Rényi entropy for the modified Exponential-Gamma (ME-G) distribution (3.1) is obtained as:

$$T_R(\gamma) = \frac{1}{1-\gamma} \log\left[\int_0^\infty \frac{\theta^{7\gamma}}{(\theta^6 + 720)^{\gamma}} (1 + x^6)^{\gamma} e^{-\theta\gamma x} dx\right]$$

$$= \frac{1}{1-\gamma} \log\left[\int_0^\infty \frac{\theta^{7\gamma}}{(\theta^6 + 720)^{\gamma}} \sum_{j=0}^{\infty} \binom{\gamma}{j} (x^6)^{j} e^{-\theta\gamma x} dx\right]$$

$$= \frac{1}{1-\gamma} \log\left[\int_0^\infty \frac{\theta^{7\gamma}}{(\theta^6 + 720)^{\gamma}} \sum_{j=0}^{\infty} \binom{\gamma}{j} (x)^{6j} e^{-\theta\gamma x} dx\right]$$

$$= \frac{1}{1-\gamma} \log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\theta^{7\gamma}}{(\theta^6 + 720)^{\gamma}} \int_0^\infty e^{-\theta\gamma x} x^{6j+1-1} dx\right]$$

$$= \frac{1}{1-\gamma} \log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\theta^{7\gamma}}{(\theta^6 + 720)^{\gamma}} \frac{\Gamma(6j+1)}{(\theta\gamma)^{6j+1}}\right]$$

$$= \frac{1}{1-\gamma} \log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\theta^{7\gamma-6j-1}}{(\theta^6 + 720)^{\gamma}} \frac{\Gamma(6j+1)}{(\gamma)^{6j+1}}\right] \quad (3.10)$$

3.2.5 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are derived as follows:

$$B(p) = \frac{1}{p\mu}\left[\int_0^{\infty} xf(x)\,dx - \int_q^{\infty} xf(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^{\infty} xf(x)\,dx\right] \tag{3.11}$$

and

$$L(p) = \frac{1}{\mu}\left[\int_0^{\infty} xf(x)\,dx - \int_q^{\infty} xf(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^{\infty} xf(x)\,dx\right] \tag{3.12}$$

respectively.

$$B(p) = \frac{1}{p\mu}\int_0^{p} F^{-1}(x)\,dx \tag{3.13}$$

and

$$L(p) = \frac{1}{\mu}\int_0^{p} F^{-1}(x)\,dx \tag{3.14}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as:

$$B = 1 - \int_0^{1} B(p)\,dp \tag{3.15}$$

and

$$G = 1 - 2\int_0^{1} L(p)\,dp \tag{3.16}$$

Respectively.

Using the PDF in equation (3.1), this gives:

$$\int_q^\infty xf(x)\,dx = \frac{\{\theta^7(q^7+q)+\theta^6(7q^6+1)+42\theta^4q^4(\theta q+5)+840\theta^2q^2(\theta q+3)+5040(\theta q+1)\}e^{\theta q}}{\theta(\theta^6+720)} \tag{3.17}$$

Now using equations (3.17) in (3.11) and (3.12), these give:

$$B(p) = \frac{1}{p}\left[1-\frac{\{\theta^7(q^7+q)+\theta^6(7q^6+1)+42\theta^4q^4(\theta q+5)+840\theta^2q^2(\theta q+3)+5040(\theta q+1)\}e^{\theta q}}{(\theta^6+5040)}\right] \tag{3.18}$$

and

$$L(p) = 1-\frac{\{\theta^7(q^7+q)+\theta^6(7q^6+1)+42\theta^4q^4(\theta q+5)+840\theta^2q^2(\theta q+3)+5040(\theta q+1)\}e^{\theta q}}{(\theta^6+5040)} \tag{3.19}$$

Now using equations (3.18) and (3.19) in equations (3.15) and (3.16), the Bonferroni and Gini indices are obtained as:

$$B = 1-\frac{\{\theta^7(q^7+q)+\theta^6(7q^6+1)+42\theta^4q^4(\theta q+5)+840\theta^2q^2(\theta q+3)+5040(\theta q+1)\}e^{\theta q}}{(\theta^6+5040)} \tag{3.20}$$

and

$$G = -1+\frac{2\{\theta^7(q^7+q)+\theta^6(7q^6+1)+42\theta^4q^4(\theta q+5)+840\theta^2q^2(\theta q+3)+5040(\theta q+1)\}e^{\theta q}}{(\theta^6+5040)} \tag{3.21}$$

3.2.6 Stress-Strength Reliability

Let Y and X be independent random variables representing stress and strength, respectively, each following the modified Exponential-Gamma (ME-G) distribution with parameters θ_1 and θ_2. Under these conditions, stress-strength reliability, denoted as R, is defined as follows:

$$R = \int_0^\infty f(x,\theta_1)\, F(x,\theta_2)\,dx$$

$$R=1-\frac{\theta_1^7\left[\begin{array}{c}479001600\theta_2^6+239500800\theta_2^5\left(\theta_1+\theta_2\right)+108864000\theta_2^4\left(\theta_1+\theta_2\right)^2+435456600\theta_2^3\\ \left(\theta_1+\theta_2\right)^3+14515200\theta_2^2\left(\theta_1+\theta_2\right)^4+3628800\theta_2\left(\theta_1+\theta_2\right)^5+720\left[\left(\theta_2^6+720\right)+\theta_2^6\right]\\ \left(\theta_1+\theta_2\right)^6+720\theta_2^5\left(\theta_1+\theta_2\right)^7+720\theta_2^4\left(\theta_1+\theta_2\right)^8+720\theta_2^3\left(\theta_1+\theta_2\right)^9+720\theta_2^2\\ \left(\theta_1+\theta_2\right)^{10}+720\theta_2\left(\theta_1+\theta_2\right)^{11}+\left(\theta_2^6+720\right)\left(\theta_1+\theta_2\right)^{12}\end{array}\right]}{\left(\theta_1^6+720\right)\left(\theta_2^6+720\right)\left(\theta_1+\theta_2\right)^{13}} \quad (3.22)$$

3.3 Parameter Estimation of the Modified Exponential-Gamma (ME-G) Distribution

3.3.1 Maximum Likelihood Estimation (MLE)

The likelihood function, L of equation (3.1), is given by:

$$L=\left(\frac{\theta^7}{\theta^6+720}\right)^n\prod_{i=1}^{n}\left(1+x_i^6\right)e^{-n\theta\bar{x}}$$

The natural log-likelihood function is thus obtained as:

$$\log(L)=n\log\left(\frac{\theta^7}{\theta^6+720}\right)+\sum_{i=1}^{n}\log\left(1+x_i^6\right)-n\theta\bar{x}$$

Now

$$\frac{d\log(L)}{d\theta}=\frac{7n}{\theta}-\frac{6n\theta^5}{(\theta^6+720)}-n\bar{x}$$

Where: $\bar{x}$ is the sample mean.

The MLE $\hat{\theta}$ of θ is the solution of the equation $\frac{d\log(L)}{d\theta}=0$ and so it can be obtained by solving the following seventh-degree polynomial equation:

$$\bar{x}\theta^7-\theta^6+720\bar{x}\theta-5040=0 \quad (3.23)$$

And further simplified into:

$$\theta^6(\bar{x}\theta-1)+720\bar{x}\theta=5040 \quad (3.24)$$

Notably, the analytic solution of (3.24) can be obtained. Therefore, the study can apply a numerical method to solve (3.24) and estimate the parameter θ. To do this, the study uses the data set pertains to aircraft window glass strength measurements, as reported by (Fuller *et al.,* 1994) study shown below: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Table 1: Parameter Estimation and Goodness of Fit Using the MLE.

$MLE(\hat{\theta})$	AIC	HQIC	BIC	CAIC
0.2271889	221.83	222.30	223.26	224.26

The results presented in Table 1 provide insights into the parameter estimation and goodness of fit of the model using the Maximum Likelihood Estimation (MLE) method. The estimated parameter suggests a reasonable fit of the modified Exponential-Gamma distribution to the given data. The computed AIC value of 221.83 indicates a good balance between model complexity and goodness of fit. The BIC value (223.26) is slightly higher, as expected, due to the additional penalty for the number of parameters. The HQIC and CAIC values, 222.30 and 224.26 respectively, further validate the model’s suitability. The consistency of these values demonstrates that the estimated parameter provides a stable and well-fitted representation of the data.

The results suggest that the modified Exponential-Gamma distribution, with the estimated parameter, adequately captures the underlying structure of the dataset while maintaining statistical efficiency.

3.3.2 Metropolis-Hastings (M-H) Algorithm

Using the M-H algorithm the study was able to simulate aircraft window glass strength measurements for n = 1000, as presented in Figure 6.

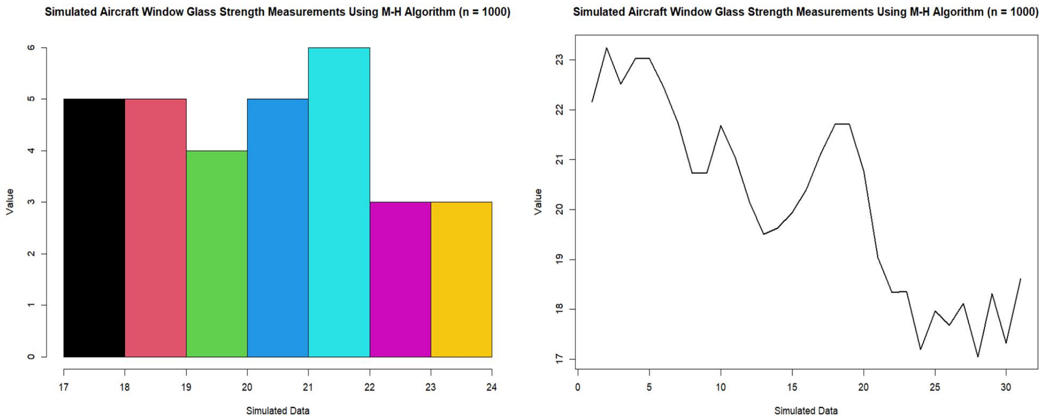

Figure 6: *A histogram and Line Plot Showing the Simulated Aircraft Window Glass Strength Measurements.*

Table 2: Parameter Estimation and Goodness of Fit Using the M-H Algorithm.

$MLE(\hat{\theta})$	AIC	HQIC	BIC	CAIC
0.3470358	187.28	187.75	188.71	189.71

The results of the parameter estimation and goodness-of-fit measures using the Metropolis-Hastings algorithm are presented in Table 2. The estimated value obtained through MLE is 0.3470358, indicating a deviation from the value estimated using direct MLE in Table 1. The lower AIC (187.28) compared to the direct MLE approach suggests that the Metropolis-Hastings method provides a better fit. Similarly, HQIC (187.75), BIC (188.71), and CAIC (189.71) values are also lower than those in Table 1, further confirming the improved model selection criteria.

The reduction in goodness-of-fit indices implies that the M-H algorithm may capture the underlying distribution dynamics more effectively than the direct MLE approach. This result highlights the advantages of employing MCMC methods in parameter estimation, particularly when dealing with complex distributions.

3.4 Comparative Analysis

Table 3: Parameter Estimation and Goodness of Fit Results of the Fitted Distributions.

Distribution	$MLE(\hat{\theta})$	AIC	HQIC	BIC	CAIC
ME-G	0.2271889	221.83	222.30	223.26	224.26
Suja	0.1622737	229.25	229.72	230.69	231.69
Rama	0.1297865	234.79	235.26	236.23	237.23
Akash	0.1000000	242.77	243.23	244.20	245.20
Lindley	0.1000000	271.50	271.97	272.93	273.93
Exponential	0.1000000	335.79	336.26	337.23	338.23

Table 3 presents the parameter estimation and goodness-of-fit metrics for six different probability distributions fitted to the data set pertains to aircraft window glass strength measurements, as reported by (Fuller *et al.,* 1994) study. The Maximum Likelihood Estimate (MLE) of the parameter (θ) varies across the models, with the ME-G distribution exhibiting the highest estimated value (0.2271889), followed by Suja (0.1622737), Rama (0.1297865), and Akash (0.1000000). The Lindley and Exponential distributions have the lowest estimated parameter values, both at 0.1000000.

Among the distributions analysed, the ME-G distribution consistently presents the lowest values for AIC (221.83), HQIC (222.30), BIC (223.26), and CAIC (224.26), suggesting that it provides the best fit to the data. The Suja distribution follows, with slightly higher AIC (229.25), HQIC (229.72), BIC (230.69), and CAIC (231.69), indicating a relatively good fit, but inferior to the ME-G distribution. The Rama and Akash distributions further increase in these criterion values, reflecting a decreasing goodness of fit. The Lindley and Exponential distributions exhibit the highest AIC, HQIC, BIC, and CAIC values, suggesting that they perform poorly in modelling the dataset compared to the other distributions.

The results suggest that the ME-G distribution provides the best fit among the tested models, as indicated by its superior goodness-of-fit measures.

Table 4: Parameter Estimation and Goodness of Fit Results of the Fitted Distributions Using Simulated Data Set Generated by M-H Algorithm.

Distribution	$MLE(\hat{\theta})$	AIC	HQIC	BIC	CAIC
ME-G	0.3470358	187.28	187.75	188.71	189.71
Suja	0.2478521	197.47	197.94	198.90	199.90
Rama	0.1981141	204.44	204.90	205.87	206.87
Akash	0.1476606	213.96	214.43	215.39	216.39
Lindley	0.1000000	229.47	229.94	230.91	231.91
Exponential	0.1000000	269.82	270.29	271.25	272.25

Table 4 presents the parameter estimates and goodness-of-fit criteria: Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), and Consistent AIC (CAIC), for six fitted distributions using a simulated data set generated using the Metropolis-Hastings (M-H) algorithm.

Among the considered models, the ME-G distribution outperformed all others across all goodness-of-fit measures, with the lowest values of AIC (187.28), HQIC (187.75), BIC (188.71), and CAIC (189.71). This indicates superior explanatory power and optimal model complexity, suggesting that the ME-G distribution is best suited to capture the underlying characteristics of the simulated data.

The Suja and Rama distributions followed, with moderately higher information criterion values, indicating comparatively weaker but still reasonable fits. In contrast, the Akash, Lindley, and Exponential distributions performed less favourably. Notably, the Exponential distribution, despite its simplicity and wide use in modelling waiting times, yielded the highest AIC (269.82) and CAIC (272.25), confirming its inadequacy for the data at hand.

Furthermore, it is worth noting that the estimated parameter values (θ) exhibit a declining trend from ME-G (0.3470) to Exponential (0.1000), reflecting differing model structures and scaling behaviours. The identical parameter estimates for Lindley and Exponential distributions (0.1000) further highlight their limited flexibility compared to more adaptable models like ME-G.

These results demonstrate that the ME-G distribution provides the best fit to the simulated data, highlighting its potential utility in contexts where flexibility and accuracy in modelling are paramount.

4 Conclusion

This study introduced the Modified Exponential-Gamma (ME-G) distribution, a novel one-parameter lifetime model derived as a mixture of an exponential and a gamma distribution with a fixed shape parameter of 7. The model was proposed to address the limitations of classical one-parameter distributions in capturing the complex behaviour of lifetime data, particularly those exhibiting non-monotonic hazard functions or heavier tails.

Through rigorous statistical analysis and empirical validation, the ME-G distribution demonstrated strong performance across multiple evaluation criteria. Parameter estimation was conducted using both the Maximum Likelihood Estimation (MLE) method and the Metropolis-Hastings (M-H) algorithm. While both approaches yielded consistent parameter estimates, the M-H algorithm outperformed the direct MLE method in terms of goodness-of-fit metrics, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and Consistent AIC (CAIC). Specifically, the M-H-based parameter estimate ($\hat{\theta}$ = 0.3470) resulted in lower AIC (187.28) and BIC (188.71) values compared to the MLE-based estimate ($\widehat{\theta}$ = 0.2272), indicating a better balance between model fit and complexity.

Comparative analysis with other widely used one-parameter lifetime models including the Suja, Rama, Akash, Lindley, and Exponential distributions further reinforced the superiority of the

ME-G distribution. In both real and simulated datasets, the ME-G consistently achieved the lowest values across all goodness-of-fit indices. The consistent underperformance of the Lindley and Exponential models, as reflected in their higher information criterion scores, highlighted their inadequacy in capturing the variability and distributional nuances of the observed data. Conversely, the ME-G distribution's flexible structure allowed it to model the data more accurately, resulting in better statistical efficiency and reliability.

These findings underscore the practical utility and robustness of the ME-G distribution, particularly in fields such as reliability engineering and survival analysis, where accurately modelling lifetime data is essential. The improved fit obtained via the Metropolis-Hastings algorithm also emphasizes the value of Bayesian simulation-based methods in estimating parameters for complex probabilistic models. Future work may extend the ME-G framework to multi-parameter forms or integrate covariate information through regression-based extensions, further enhancing its applicability to real-world data analysis scenarios.

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Declaration of Interest

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